Magnetic Moment of Constituent Fermions in Strongly Interacting Matter

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Abstract

We investigate the magnetic moment operator for constituent fermion masses for chirally symmetric theories. Constituent fermion masses are generated through a yukawa interaction of the fermion with a scalar (or /and psuedoscalar) field via the vacuum expectation value (VEV) of the scalar (or and psuedoscalar) field.

We especially consider the high baryon density π_0 condensed phase, in which chiral symmetry is spontaneously broken, with space varying expectation values of the σ and π_0 fields. This phase has a spin polarized fermi sea as the ground state. We show that there is indeed generated a macroscopic magnetization in this phase, contrary to what one would have found, if one just used a primitive phenomenological magnetic moment formula for explicit/ current fermion masses.

Furthermore, this analysis reveals that the magnetization of this state goes up as the VEV, that determines the 'mass', comes down with increasing baryon density. The consequent high magnetic field that is generated will destabilize this state at a threshold density. This is important in the context of neutron stars, as such a high density state may be responsible for very high magnetic fields in the dense core of neutron stars. This could potentially be the origin of magnetars - the stars with the largest magnetic fields in the universe.

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1 Introduction

Constituent masses are generated via yukawa interactions of the fermion with a scalar (or /and psuedoscalar) field via the vacuum expectation value (VEV) of the scalar (or and psuedoscalar) field. Constituent fermions arise in a variety of models from the Standard model of the electroweak interactions via the Higgs VEV or in the strong interactions(in the chiral symmetry limit) where the nucleon / quark masses are generated by the VEV of the sigma-field $< \sigma >$. Constituent masses are different from explicit or current masses. The obvious difference is that under a chiral symmetry transformation the exact chiral symmetric Hamiltonian remains invariant (constituent masses) whereas any current mass in the Hamiltonian explicitly breaks the chiral symmetry.

Also, there are physical situations in which the VEV's of the scalar (or and psuedoscalar) field and therefore constituent masses can vary, like at finite density or temperature, which underlines the difference with current masses which 'do not' change with finite density or temperature. Indeed, there are situations where the VEV's can even become space varying, making the difference with current masses even more dramatic. In such a context the use of the usual phenomenological formulae(e.g. for magnetic moment), that do not take account of this, can be misleading.

One such case that we will consider is the case of a high baryon density ground state for strongly interacting nucleons, which has a π_0 condensate that is a (space varying) stationary wave [Dautry and Nyman 1979, Baym 1977,Kutschera, Broniowski and Kotlorz 1990, Soni and Bhattacharya 2006]. In this case when we calculate the magnetic moment of the nucleons in the presence of a π_0 condensate using the naive formula, we find a somewhat strange result: that for a spin polarised neutron ground state, the magnetic moment vanishes when averaged over space[1]. However, when we compute the magnetic moment operator from first principles, from the chiral lagrangian, we find it has the right space dependence, which cancels out with the space dependence of the ground state to give a magnetic moment that is proportional to the total spin. Furthermore, this reveals that the magnetization of this state goes up as the VEV, that determines the 'mass', comes down with increasing baryon density..

This is important in the context of neutron stars, as such a high density ground state may be responsible for very high magnetic fields in the dense core and could be the origin of magnetars [Bhattacharya and Soni 2007,Haridass & Soni 2010]- the stars with the largest magnetic fields in the universe.

[1]R. F. Sawyer private communication

2 Current and Constituent Masses

2.1 QED

Quantum Electrodynamics is a theory with an explicit or current electron mass, m_e, which breaks chiral symmetry explicitly. The case of the electron magnetic moment operator is explicitly worked out in the text of Sakurai (Advanced Quantum Mechanics) via the Gordon decomposition (see (3-3 page 85) [Sakurai]).

$$H_{mag} = -\frac{e}{2m_e}\frac{1}{2}F_{\mu\nu}\bar{\psi}\sigma_{\mu\nu}\psi. \tag{1}$$

($\hbar = c = 1$)

2.2 The Chiral symmetric Gellmann Levy sigma model

In a chiral symmetric theory, the mass of the nucleon/quark comes from the VEV of $< \sigma >$, for example in the linear σ-model of Gellman and Levy[Gellmann Levy 1960]. In the case of exact chiral symmetry we have the following lagrangian when we couple the Gellmann Levy sigma model to the electromagnetic field

$$L = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \sum \overline{\psi}\left(\not{D} + g_y(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\right)\psi - \frac{1}{2}(\partial\mu\sigma)^2 - \frac{1}{2}(\partial\mu\vec{\pi})^2 - \frac{\lambda^2}{4}(\sigma^2 + \vec{\pi}^2 - (F)^2)^2 \tag{2}$$

In the limiting case of a small explicit pion mass being set to zero, the usual (uniform in space) symmetry breaking that follows on the minimization of the potentials above is, $< \sigma >= F = f_\pi$,and $< \vec{\pi} >= 0$.

The masses of the scalar (PS) and fermions are given by

$$< \sigma >^2 = F^2 \tag{3}$$

where F is the pion decay constant. It follows that

$$m_\sigma^2 = 2\lambda^2 F^2 \qquad m = g < \sigma >= gF \tag{4}$$

The quark or nucleon mass m, is a spontaneous mass that is generated from the spatially uniform VEV. In this case the usual magnetic moment formula above (1) works.

On the other hand when the VEV's of $< \sigma >$ and $< \vec{\pi} >$ depend on space coordinates, the above expression for the magnetic moment will not work as we will find in the following section.

3 The π_0 condensation : space dependent VEV's

3.1 The π_0 condensed ground state

Here we shall consider another realization of the expectation value of $< \sigma >$ and $< \vec{\pi} >$ corresponding to π_0 condensation. This phenomenon was first considered in the context of nuclear matter [Dautry and Nyman 1979, Baym 1977]. Such a phenomenon also occurs with our quark based chiral σ model and was considered at the mean field level by Kutschera, Broniowski and Kotlorz for the 2 flavour case [Kutschera, Broniowski and Kotlorz 1990] and by one of us for the 3 flavour case [Soni and Bhattacharya 2006]. Working in the chiral limit they found that the pion condensed state has lower energy than the uniform symmetry breaking state we have just considered for all density. This is expected, as the ansatz for the pion condenced phase is more general.

The expectation values now carry a particular space dependence

$$
\begin{aligned}
< \sigma > &= F \cos{(\vec{q}.\vec{r})} && (5)\\
< \pi_3 > &= F \sin{(\vec{q}.\vec{r})} && (6)\\
< \pi_1 > &= 0 && (7)\\
< \pi_2 > &= 0 && (8)
\end{aligned}
$$

Note that the relation, $< \sigma^2 > + < \vec{\pi}^2 >= F^2$, is preserved under this pattern of symmetry breaking. Also, when $|\vec{q}|$ goes to zero, we recover the usual space uniform phase above.

The Dirac Equation in this background is solved in [Dautry and Nyman 1979, Kutschera, Broniowski and Kotlorz 1990] by the artifact of writing the wavefunction, ψ, in terms of a chirally rotated wave function, $\chi(k)$,

$$
\psi = \exp(-i(\tau_3/2)\gamma_5 \vec{q} \cdot \vec{r}) \cdot \chi(k) \tag{9}
$$

where, $\chi(k)$ are momentum eigenfunctions.

The Hamiltonian reduces to

$$
H\chi(k) = (\vec{\alpha}.\vec{k} - \frac{1}{2}\vec{q}.\vec{\alpha}\gamma_5\tau_3 + \beta m)\chi(k) = E(k)\chi(k) \tag{10}
$$

where $m = g\sqrt{< \sigma >^2 + < \vec{\pi} >^2} = gF$

The second term arises from the condensate and has been written in terms of the relativistic spin operator, $\vec{\alpha}\gamma_5$. It is evident that if spin is parallel to $\vec{q}$ and $\tau_3 = +1$ (proton/up quark) this term is negative and if $\tau_3 = -1$ (neutron/down quark) it is positive. For spin antiparallel to $\vec{q}$ the signs of this term for $\tau_3 = +1$ and -1 are reversed.

The spectrum for the hamiltonian is the quasi particle spectrum and can be found

to be [Dautry and Nyman 1979, Kutschera, Broniowski and Kotlorz 1990]

$$E_{(-)}(k) \;=\; \sqrt{m^2+k^2+\frac{1}{4}q^2-\sqrt{m^2q^2+(\vec{q}.\vec{k})^2}} \tag{11}$$

$$E_{(+)}(k) \;=\; \sqrt{m^2+k^2+\frac{1}{4}q^2+\sqrt{m^2q^2+(\vec{q}.\vec{k})^2}} \tag{12}$$

The lower energy eigenvalue $E_{(-)}$ has spin along $\vec{q}$ for $\tau_3 = 1$, or has spin opposite to $\vec{q}$ for $\tau_3 = -1$. The higher energy eigenvalue $E_{(+)}$ has spin along $\vec{q}$ for $\tau_3 = -1$, or has spin opposite to $\vec{q}$ for $\tau_3 = +1$.

The ground state is constructed by occupying all the lower energy states $E_{(-)}$ as the gap between the lower and higher states is large. In this background the fermi sea is no longer degenerate in spin but gets polarized into the states above.

3.2 The naive magnetic moment calculation in the π_0 condensed state

Let the condensate wave vector $\vec{q}$, define the $\hat{z}$ axis. For neutrons $\tau_3 = -1$. Since the spins are aligned opposite to $\vec{q}$, we expect that there should be a net magnetic moment opposite to the $\vec{q}$ axis . Let us look at the source of the magnetic field. from the 'usual' Gordon decomposition of the electro magnetic current [Sakurai].

$$H_{mag} = \frac{e}{2m} F_{\mu\nu} \cdot \overline{\psi}\,(\sigma_{\mu,\nu}\tau_3)\,\psi \tag{13}$$

where, $m = g\sqrt{<\sigma>^2 + <\vec{\pi}>^2}$.

Substituting for $\psi = \exp(-i(\tau_3/2)\gamma_5\vec{q}\cdot\vec{r})\cdot\chi(k)$, where, $\chi(k)$ are momentum eigenfunctions, we find

$$H_{mag} = \frac{e}{2m} F_{\mu\nu} \cdot \overline{\chi(k)}\sigma_{\mu,\nu}\tau_3 \exp(-i\tau_3\gamma_5\vec{q}\cdot\vec{r})\chi(k) \tag{14}$$

The ground state is constructed by filling the spin polarized $E_{-}(k)$ states up to the fermi momentum. The total spin of the ground state is the sum over the spin polarized Fermi-sea. Do the magnetic moments also add up like the spins?

If we simply used the normal(explicit current mass) formula (14) we would get a space average over the above $\sin\vec{q}\cdot\vec{x}$ and $\cos\vec{q}\cdot\vec{x}$ terms that goes to zero.

3.3 The full chiral magnetic moment calculation in the π_0 condensed state

Using the notation of Sakurai [Sakurai], $\gamma_\mu\partial_\mu = \gamma_i\partial_i + \gamma_4\partial_4 = \gamma_i\partial_i + (i\gamma_0)(-i\partial_0)$ we may write the following chiral Dirac equations for the the π_0 condensed state

$$\big(\gamma_\mu \partial_\mu - ie\gamma_\mu A_\mu + g(\sigma + i\gamma_5 \vec{\tau}\cdot\vec{\pi})\big)\psi = 0 \tag{15}$$

$$\big(\gamma_\mu \partial_\mu - ie\gamma_\mu A_\mu + m\exp(i\tau_3\gamma_5 \vec{q}\cdot\vec{r})\big)\psi = 0 \tag{16}$$

Now we can invert this equation to get

$$\psi = -\frac{1}{m}\exp(-i\tau_3\gamma_5\vec{q}\cdot\vec{r})(\gamma_\mu\partial_\mu - ie\gamma_\mu A_\mu)\psi \tag{17}$$

Similarly, by complex conjugation with a little algebra we find

$$\overline{\psi} = \frac{1}{m}(\partial_\mu\overline{\psi}\gamma_\mu + ie\overline{\psi}\gamma_\mu A_\mu))\exp(-i\tau_3\gamma_5\vec{q}\cdot\vec{r}) \tag{18}$$

Following the 'Sakurai' trick [Sakurai], we use the ploy of dividing the electro magnetic current operator into two identical parts

$$j_\mu = \frac{e}{2}[\overline{\psi}\gamma_\mu\psi + \overline{\psi}\gamma_\mu\psi] \tag{19}$$

We then use the chiral Dirac equation(17) above for $\overline{\psi}$, and insert it in the first term and (18) for , ψ, and insert it in the second term.

After doing the Gordon decomposition, we get the following expression for the electromagnetic current operator (if we leave out all the terms that depend on the vector potential, A_μ),

$$\begin{aligned} j_\mu = \frac{e}{2m}&([(\partial_\nu\overline{\psi})\exp(i\tau_3\gamma_5\vec{q}\cdot\vec{r})\sigma_{\mu,\nu}\tau_3\psi + \overline{\psi}\exp(i\tau_3\gamma_5\vec{q}\cdot\vec{r})\sigma_{\mu,\nu}\tau_3(\partial_\nu\psi)] \\ &+ i[(\partial_\mu\overline{\psi})\exp(i\tau_3\gamma_5\vec{q}\cdot\vec{r})\tau_3\psi - \overline{\psi}\exp(i\tau_3\gamma_5\vec{q}\cdot\vec{r})\tau_3(\partial_\mu\psi)]) \end{aligned} \tag{20}$$

Notice the presence of the phase factor in the current operator. The expression reduces to the usual operator, if we put $q = 0$,for then the phase factor drops out $(= 1)$. The first part has the Lorentz structure of the magnetic moment operator and the second term reduces to a current density in the Schroedinger theory [Sakurai]. For our present purpose we will concentrate on the first term which is the magnetic moment type term.

$$\begin{aligned} H_{mag} = -j_\mu^{mag}A_\mu = -\frac{e}{2m}&A_\mu[\partial_\nu(\overline{\psi}\exp(i\tau_3\gamma_5\vec{q}\cdot\vec{r})\sigma_{\mu,\nu}\tau_3\psi) \\ &- \overline{\psi}(iq_\nu\gamma_5\tau_3)\exp(i\tau_3\gamma_5\vec{q}\cdot\vec{r})\sigma_{\mu,\nu}\tau_3\psi] \end{aligned} \tag{21}$$

By partial integration *of the first term* above, we can move the derivative to act on the vector potential , A_μ, and write it in the usual form of the magnetic moment term. However, now there is an extra term, the second term, *which needs a physical interpretation.*

The first term
The first term is the regular magnetic moment term and is

$$H_{1mag} = \frac{e}{4m} F_{\mu,\nu} [\overline{\psi} \exp(i\tau_3 \gamma_5 \vec{q} \cdot \vec{r}) \sigma_{\mu,\nu} \tau_3 \psi] \tag{22}$$

On substituting for $\psi = \exp(-i(\tau_3/2)\gamma_5 \vec{q} \cdot \vec{r}) \cdot \chi(k)$ we find that the two phase factors, from the magnetic moment operator and the wavefunction, nicely cancel to give

$$\frac{e}{2m} F_{\mu\nu} \cdot \overline{\chi(k)} \sigma_{\mu,\nu} \chi(k) \tag{23}$$

This magnetic moment is *not* space dependent and does not average to zero and thus restores the proportionality between the total spin and total magnetic moment for the ground state.

The second term

$$H_{2mag} = -(\frac{e}{2m}) A_\mu [-\overline{\psi}(iq_\nu \gamma_5 \tau_3) \exp(i\tau_3 \gamma_5 \vec{q} \cdot \vec{r}) \sigma_{\mu,\nu} \tau_3 \psi] \tag{24}$$

By using the field equations above for ψ and $\overline{\psi}$ and the 'Sakurai trick' [Sakurai] the second term can be recast to yield many terms.

We drop the terms that are not relevant for an effective magnetic moment term for reasons of economy. We also drop terms which are quadratic or higher order in 'q' and /or A_μ. We are then left with only a single term, which has the form of a magnetic moment operator. Below we write this term in terms of χ:

$$(i\frac{e}{8m^2}) F_{\mu\lambda} \cdot \overline{\chi(k)} q_\nu \gamma_5 \tau_3 \sigma_{\mu,\nu} \gamma_\lambda \chi(k) \tag{25}$$

This term is a somewhat different Dirac bilinear that interacts with the magnetic field.We note in passing that the electromagnetic interaction is not invariant under chiral transformations.

4 The magnetic field in the π_0 condensation phase transition as a function of baryon density

The interesting question that this poses in the context of the π_0 condensation phase transition is how does the chiral mass term behave as a function of baryon density? It can be seen from fig 4 [Kutschera, Broniowski and Kotlorz 1990] that as the density goes up the value of the chiral mass, $m = g\sqrt{<\sigma>^2 + <\vec{\pi}>^2}$, decreases monotonically. As this mass appears in the denominator of the expression for the magnetic moment,

the magnetic moment keeps going up as the density increases. This implies that the π_0 condensed state in which spins and magnetic moments are aligned would have higher magnetic moments and therefore higher magnetic fields as the density goes up.

Will the effect keep increasing as density goes up or is there a cap on the magnetic field? As baryon density is increased the chiral mass parameter goes down, lowering the condensate energy but simultaneously increasing the magnetic energy. An upper bound to the magnitude of the allowed magnetic field can be obtained by equating the condensation energy density to the magnetic energy density

$$(\lambda^2/4) \cdot (\frac{m}{g})^4 \simeq B^2/(8\pi) \tag{26}$$

Kutschera et al [Kutschera, Broniowski and Kotlorz 1990] use a sigma particle mass of $1200Mev$ corresponding to $\lambda^2 \simeq 75$. For a more realistic sigma particle mass of $600Mev$ we have $\lambda^2 \simeq 20$. From the figure 4 in [Kutschera, Broniowski and Kotlorz 1990] the value of the chiral mass parameter at a baryon density of $\simeq 1/F^3$ is about half its value at 0 density ($m = gf_\pi$). This yields an upper bound for the magnetic field that the condensate can sustain, $\simeq 10^{17}$ gauss, which is close to the field that we found $\simeq 10^{16}$ gauss [Bhattacharya and Soni 2007, Haridass & Soni 2010], as the core field in our magnetar model. The implication is that in the π_0 condensed state in the core of the star we are close to upper critical field. Beyond this density/magnetic field the ground state is not stable and is likely to transform into a charged pion condensate or unpolarized fermi sea (which carry no magnetic moment) - an interesting inference.

5 Acknowledgement

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